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Bayesian analysis of growth using stochastic frontier model

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Abstract

We employ Bayesian approach to the analysis of economic growth in Poland.

The results of estimation of a stochastic frontier model applied to production function of Polish voivodships in 2000 - 2004 are presented. Stochastic frontier approach allows to decompose growth into technological change, input change and efficiency change. In order to compute the posterior characteristics of the growth components we employ the Gibbs MCMC sampler.

Keywords: Bayesian analysis, Gibbs sampler, economic growth, stochastic frontier analysis

JEL codes: C11, C33, O49

1 Introduction

Economic growth and its sources belong to the most important issues of economics [Barro, Sala-i-Martin 1999], [Koop, Osiewalski, Steel 2000]. The stochastic frontier model provides a formal framework to decompose the economic growth into three components: input change, technical change and efficiency change. The aim of this paper is to present the application of the Bayesian framework in the analysis of economic growth. We model the growth by means of a stochastic production function, thus we use the sold industrial production as a measure of it. The analysis is performed for 16 voivodships in Poland in the period 2000 - 2004.

To our analysis we apply a Bayesian framework with recently developed numerical Markov Chain Monte Carlo methods. Bayesian approach seems to be the appropriate tool since it allows us to focus of any quantity of interest by deriving its posterior distribution (in particular the components of the output growth), to integrate out all nuisance parameters, to handle all restrictions and regularity conditions that result from the economic theory, as well as to deal with a large number of parameters in the model.

This research is based mainly on the papers by G. Koop, J. Osiewalski and M.F.J. Steel [1999, 2000] and to a lesser extent on [Osiewalski 2001] and [Osiewalski, Steel 1998]. In the next section we describe the stochastic frontier model. Section 3 presents the decomposition of growth, fourth section provides the model in a Bayesian framework with the derivation of Gibbs sampler used to estimate the unknown parameters. In Section 5 the results follow, last two sections present conclusions, model extensions and a summary.

2 Stochastic frontier model

The stochastic frontier model was originally proposed by W. Meeusen and J. van den Broeck [1977] and D. Aigner, C.A.K. Lovell and P. Schmidt [1977]. The model consists of the microeconomic production function and two error components: one reflecting randomness of the frontier itself and one that measures inefficiency.

In the model we assume that all comparable agents or units (herein we consider all 16 voivodships in Poland) produce according to a common technology. This assumption allows us to construct a common production function and to interpret all systematic deviations from it being a result of the underusage of inputs. In other words, the voivodship can operate either on or within a frontier. voivodships are territorial units (provinces) with full freedom of information flow thus the assumption that technology used by companies, e.g. in Podkarpackie voivodship can be copied and used by a plant operating in Dolnośląskie seems to be reasonable[1].

The following presents the production function under consideration

$$Y_{it} = f_t(K_{it}, L_{it})\tau_{it}z_{it}, \tag{2.1}$$

where Y_{it} is an output value, $f_t(K_{it}, L_{it})$ is a production function with capital and labour inputs respectively, τ_{it} $(0 < \tau_{it} \leq 1)$ is a random term reflecting production efficiency (so-called efficiency indicator) and z_{it} is a random term that captures the general stochastic nature of economic variables (which is a result of e.g. measurement error). Subscript $i = 1, \ldots, N$ identifies the producing units (in our case voivodships) in time $t = 1, \ldots, T$. The random terms are independent of each other, across time and provinces.

In our analysis we use a translog production function. This form is adopted by Koop et al. [Koop, Osiewalski, Steel 1999] in their GDP growth analysis of

[1] We are neglecting all technologies that are under patent protection.

a set of the OECD countries. Translog form allows us to reflect the variation of the data in a better way than the Cobb-Douglas function, which is one of the special cases of the translog form. Moreover, we assume that our production function is time-dependent and we will consider two cases, with and without structure imposed.

In a general case the translog production frontier can be written

$$y_{it} = x_{it}'\beta_t - u_{it} + v_{it}, \tag{2.2}$$

where

$$u_{it} = -\log(\tau_{it})$$

is a non-negative random variable,

$$v_{it} = \log(z_{it})$$

is a symmetrically distributed random variable with mean zero,

$$x_{it} = \Big(1, \quad k_{it}, \quad l_{it}, \quad l_{it} \cdot k_{it}, \quad k_{it}^2, \quad l_{it}^2, \Big)',$$

and a vector of parameters

$$\beta_t = (\beta_{t0}, \dots, \beta_{t5})'.$$

Lower case letters y, l, k indicate logarithms of the economic variables (e.g. $y = \log(Y)$). We impose non-negativity constraints on labour and capital elasticities

$$\begin{aligned} \frac{\partial y_{it}}{\partial l_{it}} &= \beta_{t2} + \beta_{t3} k_{it} + 2\beta_{t5} l_{it} \geq 0, \\ \frac{\partial y_{it}}{\partial k_{it}} &= \beta_{t1} + \beta_{t3} l_{it} + 2\beta_{t4} k_{it} \geq 0 \end{aligned} \tag{2.3}$$

for all i and t. As a local measure of economies of scale (which is also employed by Koop et al. [1999]) we use the elasticity of returns to scale ($ERTS$, see

[Varian 1992]) which, for a translog production function, is given by[2]

$$ERTS_{it} \equiv \beta_{t1} + \beta_{t2} + (\beta_{t3} + 2\beta_{t4})l_{it} + (\beta_{t3} + 2\beta_{t5})k_{it}. \qquad (2.4)$$

Thus we obtain the constant returns to scale by imposing the restriction $\beta_{t1} + \beta_{t2} = 1$, $\beta_{t4} = \beta_{t5}$ and $\beta_{t3} = -2\beta_{t4}$. The translog function will reduce to the Cobb-Douglas form for $\beta_{t3} = \beta_{t4} = \beta_{t5} = 0$.

In this paper we consider two models originally proposed by Koop et al. [1999]. In both of them we assume normally distributed random error v_{it} with a variance that remains constant across time and voivodships, $v_{it} \sim \mathcal{N}(0, \sigma^2)$. For the random efficiency indicator, $u_{it} = -\log(\tau_{it})$, we assume the exponential distribution with its expected value constant across time and voivodships, $u_{it} \sim \text{Exp}(\lambda, 0)$ (for justification of the exponential form see [Ritter, Simar 1997], [Koop, Osiewalski, Steel 1999]; for definitions see Appendix A.1).

As far as the production function parameters are concerned (there are $J = 6$ of them), we consider two versions. We define the following $(NT \times 1)$ vectors:

$$\begin{aligned} y &= (y_1', \ldots, y_t', \ldots, y_T')' \\ u &= (u_1', \ldots, u_t', \ldots, u_T')' \\ v &= (v_1, \ldots, v_t', \ldots, v_T)', \end{aligned} \qquad (2.5)$$

where $y_t = (y_{1t}, \ldots, y_{Nt})$, $u_t = (u_{1t}, \ldots, u_{Nt})$ and $v_t = (v_{1t}, \ldots, v_{Nt})$ are $(1 \times N)$ vectors and a $(N \times J)$ matrix is given by

$$X_t = (x_{1t}, \ldots, x_{Nt})'. \qquad (2.6)$$

[2]Given the production function $y = f(x)$ and a scalar $t > 0$ consider the function $y(t) = f(tx)$. Then the elasticity of returns to scale is defined as

$$e(x) = \frac{dy(t)}{dt} \cdot \frac{t}{y(t)}\bigg|_{t=1}.$$

In the first version (indicated as A version hereafter) we assume that the parameters are independent of each other for every t, hence our estimation object is a $(TJ \times 1)$ vector

$$\beta^A = (\beta_1', \ldots, \beta_T')'. \tag{2.7}$$

The model can be written as

$$y = X^A \beta^A - u + v, \tag{2.8}$$

where y, u and v are vectors defined in 2.5, β is defined as in 2.7 and X is the $(NT \times TJ)$ matrix

$$X^A = \begin{bmatrix} X_1 & & & & \\ & \ddots & & & \\ & & X_t & & \\ & & & \ddots & \\ & & & & X_T \end{bmatrix}, \tag{2.9}$$

where X_t is the matrix given in 2.6.

In the second version of our model (indicated as B) we impose a linear trend restriction on the production function parameters

$$\beta_t = \beta^* + t\beta^{**}. \tag{2.10}$$

This model can be written as in equation 2.8

$$y = X^B \beta^B - u + v, \tag{2.11}$$

where $\beta^B = ((\beta^*)' \ (\beta^{**})')'$ is a $(2J \times 1)$ vector and

$$X^B = \begin{bmatrix} X_1 & X_1 \\ \vdots & \vdots \\ X_t & tX_t \\ \vdots & \vdots \\ X_T & TX_T \end{bmatrix}$$

is the $(NT \times 2J)$ matrix. X_t is given in 2.6.

Hence to describe the production frontier in the A version we use $(NT + TJ + 2)$ parameters, whereas the dimension of the version B of the model is $(NT + 2J + 2)$.

3 Growth decomposition

The production frontier allows us to decompose the economic growth into three components: efficiency change[3], input change and technical change. Assume that the production frontiers for all producing units in periods t and $t+1$ and all inputs in these periods are given. Then the expected value of growth (further abbreviated to expected or predicted growth) can be written as

$$\mathrm{E}(y_{i,t+1} - y_{i,t}) = (x'_{i,t+1}\beta_{t+1} - x'_{i,t}\beta_t) + (u_{ti} - u_{i,t+1}). \qquad (3.12)$$

The first expression in brackets indicates technical and input change whereas the second one is the efficiency change. Let us write the first expression as

$$x'_{i,t+1}\beta_{t+1} - x'_{i,t}\beta_t = \frac{1}{2}(x_{i,t+1} - x_{i,t})'(\beta_{t+1} + \beta_t) + \frac{1}{2}(x_{i,t+1} + x_{i,t})'(\beta_{t+1} - \beta_t). \qquad (3.13)$$

The first component of equation 3.13 is the change in inputs for the average technology. The second term indicates technical progress given average inputs in two periods considered. Technical change for the ith voivodship can be measured as $\exp[x'_{*i}(\beta_{t+1} - \beta_t)]$ for a given vector of inputs x_{*i}. Due to the fact that inputs vary over time, Koop et al. [1999, 2000] propose to measure the impact of global technical change on the productivity of the ith voivodship

[3] As far as the efficiency of the production process is concerned we can differentiate between technical efficiency that results from the proper use of the production technology given inputs and allocative efficiency which is a consequence of the proper allocation of inputs (see [Osiewalski 2001]). The subject matter of the stochastic frontier model is technical efficiency.

by the geometric mean of x_{it} and $x_{i,t+1}$. Hence we obtain the technical change $TC_{i,t+1}$ given by

$$TC_{i,t+1} = \exp\left[\frac{1}{2}(x_{i,t+1} + x_{it})'(\beta_{t+1} - \beta_t)\right] \tag{3.14}$$

and the change in inputs $IC_{i,t+1}$

$$IC_{i,t+1} = \exp\left[\frac{1}{2}(x_{i,t+1} - x_{it})'(\beta_{t+1} + \beta_t)\right]. \tag{3.15}$$

The entire productivity change of the ith voivodship can be calculated as

$$PC_{i,t+1} = TC_{i,t+1} \times IC_{i,t+1}. \tag{3.16}$$

The third growth component is the efficiency change $EC_{i,t+1}$ that can be written as

$$EC_{i,t+1} = \exp[(u_{i,t} - u_{i,t+1})] = \frac{\tau_{i,t+1}}{\tau_{i,t}}. \tag{3.17}$$

Cumulated technical, input and efficiency changes for all periods considered are given by

$$CTC_i = \prod_{t=1}^{T-1} TC_{i,t+1}, \tag{3.18}$$

$$CIC_i = \prod_{t=1}^{T-1} IC_{i,t+1}, \tag{3.19}$$

and

$$CEC_i = \prod_{t=1}^{T-1} EC_{i,t+1} = \exp(u_{i,1} - u_{i,T}). \tag{3.20}$$

The average change in all periods are calculated as a geometric mean of all annual changes. Thus, for technical change we can write

$$ATC_i = (CTC_i)^{\frac{1}{T-1}}, \tag{3.21}$$

and for efficiency and input change we have

$$AEC_i = (CEC_i)^{\frac{1}{T-1}}, \tag{3.22}$$

$$AIC_i = (CIC_i)^{\frac{1}{T-1}} \tag{3.23}$$

respectively. The expected (predicted) average annual growth can be derived from equation 3.12

$$AGC_i = ATC_i \times AIC_i \times AEC_i, \tag{3.24}$$

whereas the average annual productivity becomes

$$APC_i = ATC_i \times AEC_i. \tag{3.25}$$

In order to facilitate interpretation, the final results are given in percentage points: $ATG = 100 \times (ATC_i - 1)$ for technical change, $AEG = 100 \times (AEC_i - 1)$ for efficiency change, etc.

4 Bayesian model

To investigate the production growth in Polish voivodships in 2000 - 2004 from models 2.8 and 2.11 we use a Bayesian framework. We have $N = 16$ voivodships and $T = 5$ periods (annual data from years 2000 - 2004). The observation vector is an output vector y with the exogenous variables matrix X given (defined as $X = \{X^A, X^B\}$ for a A or B version of the model):

$$y = (y; X).$$

Vector of parameters that are the object of inference is given by

$$\theta = \left(\beta' \ \sigma^2 \ \lambda \ u\right),$$

where β is defined (depending on the model version) as[4] β^A or β^B. Given the above mentioned notation we define the likelihood function as[5]

$$L(y|\theta, X) = f_N^{TN}(y|X\beta - u, \sigma_{TN}^2 I_{TN}). \tag{4.26}$$

The prior density for the vector of production function parameters $p(\beta)$ is an (improper) truncated uniform distribution, which takes the value $p(\beta) = 1$ when the regularity conditions given in 2.3 are satisfied and $p(\beta) = 0$ otherwise. For the posterior density to be well-defined our prior densities of parameters σ^2 and λ must be proper (informative), otherwise the posterior density is not σ-complete[6] (see [Fernandez, Osiewalski, Steel 1997]). In the model we use the following prior densities for σ^2 and λ

$$p(\sigma^{-2}) = (\sigma^{-2})^{n_0/2-1} \exp -\frac{a_0}{2\sigma^2}, \tag{4.27}$$

and

$$p(\lambda^{-1}) = f_G(\lambda^{-1}|\lambda_{01}, \lambda_{02}) \tag{4.28}$$

with predefined hyperparameters a_0, n_0, λ_{01} and λ_{02}. Koop et al. [1999] take $a_0 > 0$, $\lambda_{01} > 0$, $\lambda_{02} > 0$ and $n_0 \geq 0$. For $n_0 = 0$ the prior is improper yet it implies a proper posterior distribution (see Fernandez et al. [1997]). Following Koop et al. [1999] we assume $a_0 = 10^{-6}$. The resulting prior density is close to the non-informative prior $p(\sigma) \propto \sigma^{-1}$ but it assigns small weights on big realizations of σ^{-2}. The hyperparameter $\lambda_{01} = 1$ was chosen for the prior

[4] Introduction of such notation should not complicate the entire reasoning since in almost all functions discussed X and β appear as the product $X\beta$, the dimension of which is equal $(NT \times 1)$ for both models (A and B). In case of a joint distribution of all parameters or a distribution of the parameter β we should remember that for model A these dimensions are equal $(NT + TJ + 2)$ and TJ respectively, whereas for B they are $(NT + 2J + 2)$ and $2J$.

[5] For definitions see Appendix A.1

[6] A measure μ defined on a space X is σ-complete iff there exists a (infinite) sequence of subsets X_1, X_2, ... summing up to X, that $\mu(X_i)$ is finite for all i.

density of λ to be flat. $\lambda_{02} = -\log(\tau^*)$ through τ^* reflects prior beliefs about the median efficiency. Koop et al. [1999] perform a sensitivity analysis of the model with respect to different values of τ^*. The results show that '(...) the model displays an impressive degree of robustness across these very different priors'. Further on in the analysis we assume $\tau^* = 0.75$.

The prior density for u is assumed to be the Gamma distribution (see Section 2)

$$\prod_{t=1}^{T}\prod_{i=1}^{N} f_G(u_{it}|1, \lambda^{-1}). \tag{4.29}$$

The joint prior density is a product of prior distributions of all parameters

$$p(\beta, \sigma^{-2}, \lambda^{-1}, u) = p(\beta)p(\gamma^{-2})p(\lambda^{-1})p(u). \tag{4.30}$$

Having defined all priors we can write the posterior density as:

$$p(\theta|y, X) \propto f_N^{TN}(y|X\beta - u, \sigma^2 I_{TN})p(\beta)p(\sigma^{-2})p(\lambda^{-1})\prod_{t=1}^{T}\prod_{i=1}^{N} f_G(u_{it}|1, \lambda^{-1}). \tag{4.31}$$

4.1 Gibbs sampler

In order to approximate our posterior density we use Gibbs sampler, the Markov Chain Monte Carlo algorithm (for details see e.g. [Tierney 1994], [Bernardo, Smith 1994], [Casella, Robert 1999], [Roberts, Rosenthal 2004]). First we specify conditional densities of all parameters. The derivations hereafter are based on the hints given by Koop et al. [1999].

To obtain the conditional density for[7] $p(\beta|u, \sigma^{-2}, \lambda^{-1}, y, X)$ note that

$$\hat{\beta} = (X'X)^{-1}X'(y+u)$$

and

$$(y+u)'X\hat{\beta} = \hat{\beta}'X'X\hat{\beta}.$$

[7] For model A it is 30- and for model B – 12-dimensional density.

Then we calculate

$$
\begin{aligned}
& p(\beta|u,\sigma^{-2},\lambda^{-1},y,X) \propto \exp\left[-\frac{(y-X\beta+u)'(y-X\beta+u)}{2\sigma^2}\right]p(\beta) \\
= & \exp\left[-\frac{(y+u)'(y+u)-2(y+u)'X\beta+\beta'X'X\beta}{2\sigma^2}\right]p(\beta) \\
= & e^{\left[-\frac{(y+u)'(y+u)-2(y+u)'X\hat{\beta}+2\hat{\beta}'X'X\hat{\beta}-2(y+u)'X\beta+\beta'X'X\beta}{2\sigma^2}\right]}p(\beta) \\
= & \exp\left[-\frac{(y+u-X\hat{\beta})'(y+u-X\hat{\beta})+(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})}{2\sigma^2}\right]p(\beta) \\
\propto & \exp\left[-\frac{(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})}{2\sigma^2}\right]p(\beta) \qquad (4.32)
\end{aligned}
$$

since $(y+u-X\hat{\beta})'(y+u-X\hat{\beta})$ is independent of β (it is a part of a normalising constant). Obviously 4.32 is a normal distribution with mean $\hat{\beta}$ and covariance matrix $\sigma^2(X'X)^{-1}$. The next conditional density is $p(\sigma^{-2}|\beta,u,\lambda^{-1},y,X)$. Denoting ι as an $(NT \times 1)$ vector of ones we calculate

$$
\begin{aligned}
& p(\sigma^{-2}|\beta,u,\lambda^{-1},y,X) \propto (\sigma^2)^{-TN/2}e^{\left[-\frac{(y-X\beta+u)'(y-X\beta+u)}{2\sigma^2}\right]}\sigma^2 e^{-\frac{10^{-6}}{2\sigma^2}} \\
= & (\sigma^{-2})^{\frac{TN}{2}-1}\exp\left[-\frac{1}{2}[10^{-6}+(y-X\beta+u)'(y-X\beta+u)]\sigma^{-2}\right]. (4.33)
\end{aligned}
$$

The distribution given in 4.33 is a Gamma distribution with a shape parameter $\frac{TN}{2}$ and scale parameter $10^{-6}+(y-X\beta+u)'(y-X\beta+u)$ (see the definition A.3 in Appendix A.1). For our inefficiency vector u the conditional density is truncated normal.

$$
\begin{aligned}
& p(u|\beta,\sigma^{-2},\lambda^{-1},y,X) \propto \\
\propto & \exp\left[-\frac{(y-X\beta+u)'(y-X\beta+u)}{2\sigma^2}\right]\prod_{t=1}^{T}\prod_{i=1}^{N}\exp(-u_{it}\lambda^{-1})I(u_{it}\geq 0) \\
= & \exp\left[-\frac{(y-X\beta+u)'(y-X\beta+u)}{2\sigma^2}\right]\exp(-u'\iota\lambda^{-1})I(u_{it}\geq 0) \\
= & \exp\left[-\frac{(y-X\beta+u)'(y-X\beta+u)+2\sigma^2u'\iota\lambda^{-1}}{2\sigma^2}\right]I(u_{it}\geq 0) \\
\propto & e^{\left[-\frac{(y-X\beta+u)'(y-X\beta+u)+2\frac{\sigma^2}{\lambda}u'\iota-2\frac{\sigma^2}{\lambda}(X\beta-y)'\iota+(\frac{\sigma^2}{\lambda})^2}{2\sigma^2}\right]}I(u_{it}\geq 0).
\end{aligned}
$$

The last transformation is a multiplication of a density function by a constant independent of u; $I(\cdot)$ being an indicator function. We obtain

$$p(u|\beta,\sigma^{-2},\lambda^{-1},y,X) \propto e^{\left[-\frac{(u-X\beta+y+\frac{\sigma^2}{\lambda}\iota)'(u-X\beta+y+\frac{\sigma^2}{\lambda}\iota)}{2\sigma^2}\right]} I(u_{it}\geq 0), \qquad (4.34)$$

which is the truncated normal density with mean $X\beta - y - \sigma^2\lambda^{-1}\iota$ and variance $I_{TN}\sigma^2$. The last conditional density of our algorithm is $p(\lambda^{-1}|\beta,\sigma^{-2},u,y,X)$.

$$\begin{aligned} p(\lambda^{-1}|\beta,\sigma^{-2},u,y,X) &\propto \lambda^{-NT}\exp(\log(\tau^*)\lambda^{-1})\exp(-u'\iota\lambda^{-1}) \\ &= \lambda^{-NT}\exp(-(u'\iota-\log(\tau^*))\lambda^{-1}), \qquad (4.35) \end{aligned}$$

which is a Gamma distribution with shape and scale parameters $1+NT$ and $u'\iota - \log(\tau^*)$ respectively.

Summarising, in order to use the Gibbs sampler in our model we simulate in sequence from all conditional densities. We assume the following initial values

- $u^{(0)} = (0,\ldots,0)'$ – it reflects full efficiency;
- $(\sigma^2)^{(0)}$ – an unbiased OLS estimator with an assumption of full efficiency (for model A it is 0.0030, for B 0.0027);
- $(\lambda^{-1})^{(0)} = 1$ – taken arbitrarily.

The scheme below presents one step of the Gibbs sampler

1. generate $\beta^{(j)}|(\sigma^{-2})^{(j-1)},(\lambda^{-1})^{(j-1)},u^{(j-1)}$ from

$$f_N^K\left(\beta^{(j)}\middle|(X'X)^{-1}X'(y+u^{(j-1)}),(\sigma^2)^{(j-1)}(X'X)^{-1}\right)p(\beta^{(j)})$$

2. generate $(\sigma^2)^{(j)}|\beta^{(j)},(\lambda^{-1})^{(j-1)},u^{(j-1)}$ from

$$f_G\left((\sigma^2)^{(j)}\middle|\frac{TN}{2},\frac{1}{2}[10^{-6}+(y-X\beta^{(j)}+u^{(j-1)})'(y-X\beta^{(j)}+u^{(j-1)})]\right)$$

3. generate $u^{(j)}|\beta^{(j)}, (\sigma^{-2})^{(j)}, (\lambda^{-1})^{(j-1)}$ from

$$f_N^{TN}\left(u^{(j)}\middle|X\beta^{(j)} - y - \frac{(\sigma^2)^{(j)}}{(\lambda)^{(j-1)}}\iota, I_{TN}(\sigma^2)^{(j)}\right) I(u_{it}^{(j)} \geq 0)$$

4. generate $(\lambda^{-1})^{(j)}|\beta^{(j)}, (\sigma^{-2})^{(j)}, u^{(j)}$ from

$$f_G\left((\lambda^{-1})^{(j)}\middle|1 + NT, -\log(\tau^*) + \iota' u^{(j)}\right).$$

The dimension of the conditional density for β denoted as K is equal to 30 and 12 for the models A and B respectively. Samples generated from this algorithm converge to marginal posterior densities (see [Koop, Osiewalski, Steel 1995], [Osiewalski, Steel 1995]).

We generate a trajectory of 102 000 samples, from which 2000 are burn-in passes (they are being discarded so as to eliminate the start-up effects). The number of burn-in passes is chosen arbitrarily[8]. More in depth discussion of Gibbs sampler applied to stochastic frontier model can be found in [Osiewalski, Steel 1995].

The Gibbs algorithm presented in this section requires sampling from a multi-dimensional truncated normal distribution. Brief description of commonly used methods is enclosed in Appendix A.2.

5 Results

A set of annual data for 16 voivodships in years 2000 - 2004 was taken from Rocznik Statystyczny Przemysłu GUS [2003, 2005]. In particular, data were collected regarding

- industrial production sold in million PLN, current prices (variable Y)
- employment in industry in thousands (variable L)

[8]Koop et al. [1999] consider this number as sufficient to ensure the convergence.

- gross capital assets in industry in million PLN, current prices (variable K).

Variables were divided by production sold price indicator (base=1 in 2000), then logarithms taken.

We obtained samples from marginal posteriors of all parameters in θ. In all tables we present the means of these samples.[9] In brackets sample standard deviations are given, yet they can be used only as a rough approximation of our 'true' standard deviation – such an estimator can be heavily biased (see the example of 'Witch's Hat' in [Roberts, Rosenthal 2004] and e.g. [Geyer 1992]), thus we avoid inference on its basis.

In Bayesian approach it is common to present the results of the model with the largest posterior probability (for the details see e.g. [Geweke 1995], [Bernardo, Smith 1994]). However, the calculation of a normalizing constant within the Gibbs sampler framework is numerically complicated ([Koop et al. 1999]). Hence, we present the results of both models A and B and we choose the better one according to the goodness of fit.

Tables 1 and 2 present the estimates of posterior expected values (in brackets the posterior standard deviations are given) of the production function parameters for models A and B respectively. These parameters, however, do not have any direct economic interpretation. Our main interest is the decomposition of the average annual production growth in the voivodships into technical change, input change and efficiency change.

Tables 3 and 4 present the posterior expected values and standard deviations of the functions of the parameters calculated according to formulae given in section 3. All results are given in percentage points. In the second columns of both tables ('production growth') the sample geometric mean of

[9] Koop et al. [1999] present means and standard deviations whereas in [Koop, Osiewalski, Steel 2000] medians and interquartile ranges of posterior distributions.

Table 1: Posterior expected values and standard deviations (in brackets) of production function parameters – model A

period t	β_{t0}	β_{t1}	β_{t2}	β_{t3}	β_{t4}	β_{t5}
1	15.6937	-11.9311	11.8882	-3.7189	2.3417	1.1812
	(7.5207)	(6.8526)	(7.0629)	(2.9896)	(1.5059)	(1.5497)
2	17.3519	-12.6255	11.9512	-3.9799	2.4716	1.4454
	(7.7172)	(6.8000)	(6.9246)	(2.8773)	(1.4597)	(1.4815)
3	18.7267	-13.3788	12.4156	-4.1569	2.5834	1.5428
	(8.3288)	(7.2940)	(7.4602)	(3.1424)	(1.5706)	(1.6323)
4	15.2755	-10.7286	10.2116	-3.5335	2.1330	1.4111
	(7.4640)	(6.5564)	(6.8607)	(3.0875)	(1.4720)	(1.6869)
5	19.4765	-14.0569	12.9469	-5.6057	3.0400	2.8452
	(9.2777)	(8.0143)	(8.0453)	(3.5402)	(1.7558)	(1.8488)

Table 2: Posterior expected values and standard deviations (in brackets) of production function parameters – model B

period t	β_{t0}	β_{t1}	β_{t2}	β_{t3}	β_{t4}	β_{t5}
1	13.2044	-9.3642	9.1454	-2.4864	1.7286	0.5933
	(4.4231)	(3.9107)	(4.0120)	(1.7008)	(0.8516)	(0.9029)
2	13.8813	-9.7095	9.2660	-2.8292	1.8517	0.9092
	(3.1793)	(2.8086)	(2.8856)	(1.2188)	(0.6102)	(0.6463)
3	14.5581	-10.0547	9.3866	-3.1721	1.9749	1.2252
	(2.9111)	(2.5418)	(2.6035)	(1.1194)	(0.5545)	(0.5920)
4	15.2350	-10.4000	9.5072	-3.5149	2.0981	1.5412
	(3.8292)	(3.3182)	(3.3838)	(1.4816)	(0.7284)	(0.7834)
5	15.9119	-10.7452	9.6277	-3.8577	2.2213	1.8571
	(5.3548)	(4.6409)	(4.7267)	(2.0765)	(1.0207)	(1.0987)

the production is given. AGG is an annual average production growth that results from the model, AIG – the average input growth, ATG – the average technical growth, AEG – the average efficiency growth, APG is the average productivity growth and the ASG is the average sample growth.[10]

The closer is the expected average production growth (AGG) to the sample growth, the better is a model (see [Koop, Osiewalski, Steel 2000]). Hence in

[10] These are the quantities calculated according to equations 3.21 – 3.25 given in percentage points.

order to choose a model, on the basis of which we will make inference about the production growth, we use the mean absolute percentage error ($MAPE$), calculated for differences between AGG and ASG. For model A the $MAPE$ was equal to 20% while for model B it was 28%. According to that criterion in further analysis we will use, the more general, model A. This approach however can be seen as an 'eclectic' one since $MAPE$ is used in a classical statistical framework as a measure of the forecast quality. A formal choice of a better model using Bayes factor is technically and numerically complex as it requires the calculation of normalizing constants which, in turn, is complicated due to improper prior distributions employed (see [Osiewalski 2001] for details).

Except for the voivodships Kujawsko-pomorskie, Śląskie, Wielkopolskie and Zachodniopomorskie, the AGG is relatively close to the sample values. ATG and AEG approximately sum up to APG. The latter one, together with AIG, yields the AGG for all provinces under consideration.

The highest predicted production growth accounted for 3.89% in Wielkopolskie and it was 0.5% greater than the respective sample mean (within the sample the largest growth was found for Śląskie with 3.69% and its expected value 3.34%). The smallest predicted growth (0.76%) was in Zachodniopomorskie (sample mean was 0.26%). The average sample growth for all voivodships was equal to 1.94% with predicted value of 1.95% that indicates that model A fits the data reasonably well (in the restricted model B the average AGG was equal to 1.80%).

Decomposition of growth into its components reveals that the major determinant of growth was the technical change (ATG). The highest (more than 4.5%) growth was denoted in Podlaskie and Opolskie, the smallest in Małopolskie where it amounted to 1.7%. The average ATG for all provinces was 2.88%. Such an impact of the technical change on the production growth can be explained by changes in the production structure, inflow of foreign investments

Table 3: Production growth components in voivodships, model A (results given in percentage points)

voivodship	production growth	AGG^A	AIG^A	ATG^A	AEG^A	APG^A
Dolnośląskie	2.55	2.50	0.67	1.79	0.04	1.82
		(1.31)	(0.29)	(1.16)	(1.33)	(1.35)
Kujawsko-pomorskie	1.65	1.16	-1.32	2.21	0.30	2.51
		(1.14)	(0.58)	(1.04)	(1.02)	(1.30)
Lubelskie	0.85	0.94	-1.45	2.59	-0.16	2.43
		(1.32)	(0.44)	(0.95)	(1.37)	(1.41)
Lubuskie	2.12	2.18	-1.59	3.84	0.01	3.84
		(1.19)	(0.75)	(1.34)	(0.97)	(1.43)
Łódzkie	0.81	1.03	-0.36	2.15	-0.74	1.39
		(1.49)	(0.36)	(1.02)	(1.62)	(1.53)
Małopolskie	0.95	0.99	-0.68	1.71	-0.03	1.68
		(1.13)	(0.32)	(1.02)	(1.02)	(1.19)
Mazowieckie	2.20	2.46	0.61	1.99	-0.12	1.85
		(1.61)	(0.79)	(2.27)	(1.14)	(1.94)
Opolskie	2.92	2.85	-1.64	4.47	0.13	4.57
		(1.66)	(0.69)	(2.90)	(1.84)	(2.05)
Podkarpackie	1.55	1.54	-0.45	2.07	-0.06	2.00
		(1.42)	(0.26)	(0.95)	(1.51)	(1.45)
Podlaskie	2.49	2.54	-1.80	4.46	-0.02	4.43
		(1.44)	(0.55)	(1.77)	(1.10)	(1.60)
Pomorskie	1.99	1.38	-0.88	2.21	0.08	2.29
		(1.00)	(0.46)	(1.01)	(0.58)	(1.09)
Śląskie	3.69	3.34	-0.76	3.43	0.71	4.13
		(1.60)	(0.49)	(2.24)	(2.02)	(1.64)
Świętokrzyskie	2.04	2.01	-1.28	3.27	0.07	3.34
		(1.30)	(0.43)	(1.43)	(1.22)	(1.42)
Warmińsko-mazurskie	1.52	1.59	-1.67	3.82	-0.42	3.32
		(1.63)	(0.89)	(2.86)	(2.65)	(1.82)
Wielkopolskie	3.38	3.89	0.55	3.65	-0.30	3.33
		(1.57)	(0.67)	(1.77)	(1.25)	(1.52)
Zachodniopomorskie	0.26	0.72	-1.37	2.35	-0.22	2.12
		(1.15)	(0.24)	(1.05)	(0.99)	(1.19)
geometric mean	1.93	1.94	-0.84	2.87	-0.05	2.81

and import of new technologies (see the industry report prepared by the Ministry of Economy and Labour [2005]). As a consequence of an increase in expenditures on the innovative activity and modernization (especially since 2003) the share of new and modernized goods in the total production sold is

Table 4: Production growth components in voivodships, model B (results given in percentage points)

voivodship	production growth	AGG^B	AIG^B	ATG^B	AEG^B	APG^B
Dolnośląskie	2.55	2.75 (1.15)	0.90 (0.22)	1.80 (0.86)	0.05 (1.06)	1.84 (1.15)
Kujawsko-pomorskie	1.65	0.93 (0.98)	-1.29 (0.37)	2.05 (0.73)	0.20 (0.86)	2.25 (1.03)
Lubelskie	0.85	1.00 (1.24)	-1.18 (0.29)	2.39 (0.71)	-0.18 (1.25)	2.20 (1.27)
Lubuskie	2.12	2.27 (1.02)	-1.05 (0.43)	3.34 (0.97)	0.02 (0.83)	3.36 (1.12)
Łódzkie	0.81	0.92 (1.37)	-0.39 (0.25)	2.06 (0.73)	-0.72 (1.43)	1.32 (1.38)
Małopolskie	0.95	0.64 (0.98)	-1.22 (0.35)	1.92 (0.77)	-0.03 (0.82)	1.88 (0.99)
Mazowieckie	2.20	2.37 (1.44)	0.26 (0.45)	2.14 (1.68)	-0.03 (0.61)	2.11 (1.61)
Opolskie	2.92	2.67 (1.43)	-1.70 (0.47)	4.36 (1.96)	0.10 (0.89)	4.46 (1.77)
Podkarpackie	1.55	1.73 (1.33)	-0.21 (0.14)	2.03 (0.74)	-0.07 (1.38)	1.95 (1.34)
Podlaskie	2.49	2.65 (1.23)	-1.31 (0.32)	3.96 (1.36)	0.06 (0.91)	4.02 (1.33)
Pomorskie	1.99	1.19 (0.81)	-0.92 (0.23)	2.07 (0.74)	0.06 (0.47)	2.13 (0.83)
Śląskie	3.69	2.10 (1.38)	-1.84 (0.38)	3.30 (1.61)	0.69 (1.44)	4.01 (1.41)
Świętokrzyskie	2.04	1.28 (1.11)	-2.09 (0.42)	3.39 (1.01)	0.06 (0.93)	3.45 (1.16)
Warmińsko-mazurskie	1.52	1.64 (1.41)	-1.16 (0.50)	3.14 (1.93)	-0.27 (2.02)	2.83 (1.47)
Wielkopolskie	3.38	4.08 (1.31)	1.10 (0.38)	3.05 (1.20)	-0.10 (0.82)	2.95 (1.16)
Zachodniopomorskie	0.26	0.63 (0.98)	-1.50 (0.15)	2.35 (0.79)	-0.19 (0.82)	2.16 (1.00)
geometric mean	1.93	1.80	-0.85	2.70	-0.02	2.67

rising. Due to the Polish accession to the European Union in May 2004 (as well as preparations to it), Polish companies have been forced towards the production of highly processed goods that can meet the standards of the EU markets, which requires usage of new technologies. An increase in share of these goods both in exports (in 2004 the share of EU markets in Polish export exceeded 80%) and in imports can be observed, that again is an incentive for Polish companies to modernize and to improve their competitiveness. Ideally, through direct foreign investments Polish entrepreneurs and managers become acquainted with new ways of management and learn modern production organization.

A negative input growth (AIG) was the main production-growth-decreasing factor in most of the provinces considered. Slight growth was experienced only by Dolnośląskie, Wielkopolskie and Mazowieckie voivodships. Negative AIG can be explained by a sharp drop of the number of employees in the industry sector caused by migration. The biggest decline was denoted in Podlaskie (-1.8%) and in Warmińsko-mazurskie (-1.67%). The average input growth across all provinces decreased by 0.84%. Further decomposition of the input growth into labour and capital changes is precluded by the form of the production function employed.

A decrease in capital and labour inputs is not compensated by an increase in efficiency (AEG). Moreover, AEG declined by 0.05 % on average across all voivodships. The greatest growth was denoted in Śląskie (by 0.7%) and Kujawskopomorskie (by 0.3%). On the other hand the biggest decline was experienced by Łódzkie (-0.7%), Warmińsko-mazurskie (-0.4%) and Wielkopolskie (-0.3%).

The efficiency changes have a minor influence on the production and productivity growth. Merely in Śląskie and Kujawsko-pomorskie AEG can be considered significant. High production growths in Śląskie and Wielkopolskie stem from high technical growth and relatively small decline in input in the first case and a slight increase in the latter. High technical growth in Opolskie and Podlaskie is diminished significantly by a relatively high decrease in inputs.

Table 5: Production efficiencies by voivodships

voivodship	2000		2004		K/L
Pomorskie	0.982	(0.017)	0.985	(0.015)	130.3
Kujawsko-pomorskie	0.958	(0.032)	0.969	(0.026)	133.7
Lubuskie	0.969	(0.027)	0.969	(0.026)	135.3
Podlaskie	0.967	(0.029)	0.966	(0.030)	132.7
Mazowieckie	0.971	(0.029)	0.966	(0.033)	236.7
Małopolskie	0.965	(0.028)	0.964	(0.029)	151.2
Zachodniopomorskie	0.972	(0.024)	0.963	(0.029)	160.6
Świętokrzyskie	0.958	(0.034)	0.960	(0.032)	193.9
Wielkopolskie	0.969	(0.030)	0.958	(0.039)	121.2
Opolskie	0.946	(0.050)	0.950	(0.046)	255.9
Dolnośląskie	0.947	(0.037)	0.948	(0.037)	176.0
Śląskie	0.923	(0.059)	0.948	(0.044)	178.8
Lubelskie	0.941	(0.039)	0.934	(0.041)	135.8
Podkarpackie	0.924	(0.045)	0.921	(0.046)	136.1
Łódzkie	0.925	(0.044)	0.898	(0.050)	144.1
Warmińsko-mazurskie	0.910	(0.069)	0.895	(0.081)	98.1
geometric mean	0.952		0.950		157.5

Table 5 presents the posterior distribution characteristics ordered by the technical efficiency indicators in years 2000 and 2004 and the average capital assets per worker ratio (K/L) for all Polish voivodships. The smallest efficiency indicators (less than 90 %) are observed in Warmińsko-mazurskie and Łódzkie voivodships. Both of them experienced decline in efficiency in comparison with the year 2000. It is remarkable that in Warmińsko-mazurskie the capital to labour ratio is the smallest, albeit, as can be observed, high K/L ratio does not automatically imply high efficiency, in Opolskie and Mazowieckie, where this ratio is relatively big, efficiency indicator is close to the average. Furthermore, the poorest five voivodships (Warmińsko-Mazurskie, Podlaskie, Lubelskie, Świętokrzyskie and Podkarpackie; measured by GDP *per capita*, [Eurostat 2004]) are characterised by the smallest efficiency indicators. Amongst the richest with respect to GDP *per capita* voivodships (Mazowieckie, Śląskie, Wielkopolskie, Dolnośląskie, Pomorskie, and Zachodniopomorskie [Eurostat 2004]) three have efficiency higher than the average, finally, in Pomorskie the efficiency indicator is the highest of all provinces (both in 2000 and 2004), which accounts for 98.5%.

6 Conclusions and model extensions

The presented model can serve as a tool for analysis of development discrepancies and convergence issues of the provinces under consideration, especially in the context of the allocation of the EU Structural Funds for the poorest regions. European Regional Development Fund (ERDF) seems to be of utmost importance, since it aims at economic and social cohesion and correction of regional imbalances. Its main tasks are provision of infrastructure development, transfer of technology, aid for investment in education as well as in small and medium-sized enterprises, etc. [Structural Funds webpage 2007].

Stochastic frontier model presented in this article is only an example of the Bayesian inference in econometrics. Koop et al. [1999, 2000] analyse different versions of the stochastic frontier production function applied to economic growth decomposition. Variance of the random error v and the parameter λ can be time-dependent, moreover, the latter of them may depend on other exogenous variables. Production function parameters need not to be constant across all producing units. This assumption can be tested or even disregarded, parameters may differ among appropriately selected groups of units (with respect to e.g. income level, geographical position, etc.), see [Koop, Osiewalski, Steel 2000].

There may be different production function forms used, as well as latent variables. In the model by Koop et al. [2000] the effective inputs of labour and capital are used, which is justified by the fact that in different producing units (in their case countries) the same outlays need not to lead to the same efficiency in production.

In our analysis we used a model with linear trend restrictions imposed on the production function parameters. This trend can be any polynomial of higher degree, however a researcher should remember that it is rational only if the number of parameters of the restricted model is smaller than the number of parameters of the unrestricted one. Production functions parameters can be assumed to be an autoregressive process (usually $AR(1)$), see [Koop, Osiewalski, Steel 2000].

It should be stressed that the stochastic frontier model is usually applied in microeconomic research, mainly concerning companies. For instance, J. Marzec [2001, 2002] and J. Marzec with J. Osiewalski [1998] apply this model to the cost-efficiency analysis of Polish banks, J. Osiewalski and A. Osiewalska [1998-1999] use it to investigate the activity of Polish libraries.

7 Summary

We presented the results of the Bayesian econometric methods applied to analysis of the production growth in Polish voivodships in years 2000-2004 using a stochastic frontier model. We used prior informative distributions with some restrictions resulting from the economic theory. Other informative priors are the approximations of the reference priors or the results are insensitive to them. Two alternative versions of the model were considered: one with no restrictions on the production function parameters and one with a linear trend structure imposed. In order to find the posterior characteristics we applied a relatively simple, based on Markov Chains, Gibbs algorithm. Simulation was carried out using a standard personal computer. An effective method of sampling from normal truncated distribution was required.

Posterior means of the production growth obtained from the model fitted the data reasonably well. These characteristics served as a tool of model selection. Furthermore we decomposed the production growth drawing the conclusion that the main driving force of growth was technical change, yet the negative input growth hampered it significantly. Efficiency indicators were of minor importance to overall growth.

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A Appendices

A.1 Definitions

We present briefly definitions of the probability density distributions used herein.

- single-variate normal distribution

$$f_N(x|\mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < +\infty, \tag{A.1}$$

  $-\infty < \mu < +\infty$, $0 < \sigma < +\infty$. Moments: $E(x) = \mu$, $Var(x) = \sigma^2$.

- multi-variate normal distribution

$$f_N^k(\mathbf{x}|\mu, \Sigma) = (2\pi)^{-\frac{k}{2}} |\Sigma|^{1/2} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\right], \quad -\infty < x_i < +\infty, \tag{A.2}$$

  where $\mathbf{x} = (x_1, \ldots, x_k)'$, $E(\mathbf{x}) = \mu = (\mu_1, \ldots, \mu_k)'$, Σ – covariance matrix, $(k \times k)$, symmetric, positive definite.

- Gamma distribution

$$f_G(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx), \quad 0 < x < +\infty, \tag{A.3}$$

  parameters $a, b > 0$. Moments: $E(x) = a/b$, $Var(x) = a/b^2$.

- exponential distribution (Gamma distribution with $a = 1$ and $b = 1/\lambda$)

$$f_{\text{Exp}}(x|\lambda, \alpha) = \lambda^{-1} \exp(-\lambda^{-1}(x-\alpha)), \quad \alpha < x < +\infty, \tag{A.4}$$

  where $\alpha \in \mathbf{R}$ is a location parameter. Moments: $E(x) = 1/\lambda$, $Var(x) = 1/\lambda^2$. The alternative notation:

$$x \sim \text{Exp}(\lambda, \alpha).$$

- uniform distribution

$$x \sim \mathcal{U}[a,b] \Leftrightarrow f_U(x|a,b) = \frac{1}{b-a}, \quad a \leq x \leq b. \tag{A.5}$$

Parameters $a, b \in \mathbf{R}$. Moments: $E(x) = (a+b)/2$, $Var(X) = (b-a)^2/12$.

A.2 Truncated normal sampling

We present shortly a selection of methods concerning sampling from the truncated normal distribution. To simplify we consider only left-truncation. Denote by $\mathcal{N}_+(\mu, a, \sigma^2)$ the left truncated normal density, the pdf of which can be written as

$$f(x|\mu, a, \sigma^2) = \frac{\exp(-(x-\mu)^2/2\sigma^2)}{\sqrt{2\pi}\sigma[1-\Phi((a-\mu)/\sigma)]} I(x \geq a), \tag{A.6}$$

where $\Phi(\cdot)$ is a standard normal cumulative distribution function (cdf).

- The simplest method is to simulate from a non-truncated normal and discard the samples from the truncation range. This method may be ineffective when $a > \mu$ or in case of sampling from a multivariate normal.

- A method proposed by e.g. Gelfand et al. [Gelfand, Smith, Lee 1992] or Chen and Deely [Chen, Deely 1992] consists of sampling from $u \sim \mathcal{U}[0,1]$ and calculating

  $$z = \mu + \sigma\Phi^{-1}\left[\Phi\left(\frac{a-\mu}{\sigma}\right) + u\left(1 - \Phi\left(\frac{a-\mu}{\sigma}\right)\right)\right], \tag{A.7}$$

  where $\Phi(\cdot)$ denotes a standard normal cdf. However Robert [1995] indicates that the cdf and its inverse approximations may be inaccurate in case the difference $a - \mu$ is significant(e.g. more than 5 standard deviations). The advantage of this algorithm is its simplicity and relative efficiency.

- Robert [1995] proposed an accept-reject algorithm. For a normal distribution $\mathcal{N}(0, a, 1),\ a > 0$ the optimal exponential algorithm consists of the following

  1. draw z from $\alpha^* \exp(-\alpha^*(z-a))I(z \geq a)$ (see the definitions in Appendix A.1), where

     $$\alpha^*(a) = \frac{a + \sqrt{a^2+4}}{2};$$

2. compute $\rho(z) = \exp[-(z-\alpha^*)^2/2]$;
3. draw $u \sim \mathcal{U}[0,1]$ and if $u \leq \rho(z)$ take $x = z$, otherwise return to step 1.

- The method proposed by Tanner and Wong [Tanner, Wong 1987] is based on the completion of the pdf into a bivariate density[11]. It allows us to simulate a Markov chain using the two-dimensional Gibbs algorithm, the subchain of which converges to our density.

  To simulate from the distribution

  $$f(x) \propto \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \ x \geq a,$$

  we complete f into $g(x,z)$:

  $$g(x,z) \propto I(x \geq a) I\left(0 \leq z \leq \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right). \tag{A.8}$$

  Conditional densities for x and z are given by

  $$g(x|z) \propto I\left(a \leq x \leq \mu + \sqrt{-2\sigma^2 \log z}\right) \tag{A.9}$$
  $$g(z|x) \propto I\left(0 \leq z \leq \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right). \tag{A.10}$$

  Gibbs algorithm consists of drawing samples by turns from

  $$x^{(j+1)}|z^{(j)} \sim \mathcal{U}\left(a, \mu + \sqrt{-2\sigma^2 \log z}\right)$$

  $$z^{(j+1)}|x^{(j+1)} \sim \mathcal{U}\left(0, \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right)$$

  for some initial values (x_0, z_0). The sample (x_n) converges to the truncated normal distribution.

[11] With a given function $f(x)$, a function $g(x,z)$ that satisfies

$$f(x) = \int_{\mathcal{Z}} g(x,z)dz$$

is called a completion of f.

- Sampling from a multivariate normal density (in case of non-diagonal covariance matrix) is based on the iterative sampling from single-variate conditional normal distributions (see e.g. [Robert 1995]). To simplify assume that we draw samples from the left truncated (in $a = (a_1, \ldots, a_k)'$) density given in equation A.2.

  Denote by $-i$ an i-th element, column or verse removed from a vector or a matrix. We sample elements of a vector $\mathbf{x}^{(j)}$ according to a scheme
  1. $x_1 \sim f_N(x_1^{(j)}|E(x_1|(x_2^{(j-1)}, \ldots, x_k^{(j-1)}), \sigma_1^2)$;
  2. $x_1 \sim f_N(x_2^{(j)}|E(x_2|(x_1^{(j)}, x_3^{(j-1)}, \ldots, x_k^{(j-1)}), \sigma_2^2)$;
  $\vdots$
  k. $x_k \sim f_N(x_k^{(j)}|E(x_k|(x_1^{(j)}, \ldots, x_{k-1}^{(j)}), \sigma_2^2)$;
  where

$$E(x_i|x_{-i}) = \mu_i + \Sigma'_{i,-i}\Sigma^{-1}_{-i,-i}(x_{-i} - \mu_{-i}),$$

$$\sigma_i^2 = \Sigma_{i,i} - \Sigma'_{i,-i}\Sigma^{-1}_{-i,-i}\Sigma_{i,-i}.$$